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Analytic Continuation of Moyal and Operator Algebras

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Abstract

We consider the analytic continuation of the commutator $[q^m, p^n]$ for values of $m, n \neq$ positive integers. This is carried out both in the Moyal algebra and the operator algebra.

Introduction

The question of the ordering of the position and momentum operators $\hat{q}$ and $\hat{p}$ in a monomial has been extensively discussed in the literature (Cohen, 1966; Margenau & Cohen, 1967). This is equivalently also a study of the 'Moyal algebra' of functions on phase space (Misra & Shankara, 1968). However, both these discussions are almost invariably (Rosenbaum, 1967) restricted to polynomials. But more often, one comes across potentials which are inverse powers of $\hat{q}$. As regards $\hat{p}$, it is of course true that they generally appear only in positive integral powers, but in the solutions of some wave equations one basically introduces fractional powers of $\hat{p}$ (Erdelyi, 1962). Hence it would be reasonable to extend the investigation of the ordering problem to cases of products which contain arbitrary powers of $\hat{q}$ and $\hat{p}$. This becomes tractable through the methods of modern operational calculus. Fractional differentiation is defined by an analytic continuation of the Cauchy formula for the nth primitive of a function (Gelfand & Shilov, 1964). The theory requires that the solutions of the equations in that case be restricted to a class of functions which fall rapidly at infinity faster than $|q|^{-m}$ for any m and also have only semi-infinite support.

The definitions of fractional differentiation will be introduced briefly in Section 1. In Section 2 we give a discussion of the Moyal products $p^\alpha \times q^\beta$ where α and β are not positive integers. In Section 3 we shall study a generating function for the generalized Laguerre polynomials that appear in Section 2. In the next section operator products are considered and finally we conclude with a study of the generalised rule of operator correspondences.

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1. *Fractional Differentiation*

The Moyal product (Mehta, 1964) of two functions A and B is defined by

$$A(p,q) \times B(p,q) = e^{-i\partial^2/\partial\tau\partial\eta} A(\eta,q)\, B(p,\tau)\Big|_{\eta=p}^{\tau=q} \tag{1.1}$$

This is associative, and only when A and B mix up p and q is it also non-commutative. Thus

$$p \times q = qp - i \tag{1.2}$$

and

$$q \times p - p \times q = i \tag{1.3}$$

which is the same as the commutator

$$\hat{q}\hat{p} - \hat{p}\hat{q} = i \tag{1.4}$$

Thus, there exists an algebra isomorphism between the Moyal product $q^m \times p^n$ and the operator product $\hat{q}^m \hat{p}^n$.

The purpose of the present work is to extend these ideas to the cases where powers of q and p are not restricted to positive integers. A general power of $\hat{p} = -id/dq$ is defined by (Gelfand & Shilov, 1964)

$$\left(\frac{d}{dq}\right)^{\alpha} \psi(q) = \frac{1}{\Gamma(-\alpha)} \int_0^q \psi(t)\,(q-t)^{-\alpha-1}\, dt \tag{1.5}$$

where α is any complex number. When it is real, this equation defines αth order differentiation for $\alpha > 0$ and αth order primitive for $\alpha < 0$. But q is a real variable. It is also possible to extend this definition to complex q (Osler, 1970) by

$$\left(\frac{d}{dq}\right)^{\alpha} \psi(q) = \frac{\Gamma(\alpha+1)}{2\pi i} \int \psi(z)\,(q-z)^{-\alpha-1}\, dz$$

where the branch cut belonging to $(q-z)^{-\alpha-1}$, which starts at q, is taken through the origin, and the contour of integration starts and ends at the origin and passes anticlockwise round q without crossing the cut and without enclosing any singularity of $\psi(q)$. This is defined for $\alpha \neq -1, -2, \ldots$ and for $\psi(q)$ of the form $q^{\beta}\phi(q)$ where $\phi(q)$ is analytic with no pole at the origin and $\beta \neq -1, -2, \ldots$. But we shall restrict the discussions to (1.5) covering the case of functions of a real variable.

It can be verified that (1.5) satisfies the following rules of 'fractional differentiation':

$$\left(\frac{d}{dq}\right)^{\alpha} e^{\lambda q} = \lambda^{\alpha} e^{\lambda q}, \tag{1.6}$$

$$\left(\frac{d}{dq}\right)^{\alpha} q^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} q^{\beta-\alpha} \tag{1.7}$$

$$\left(\frac{d}{dq}\right)^{\alpha} \left(\frac{d}{dq}\right)^{\beta} q^{\gamma} = \left(\frac{d}{dq}\right)^{\alpha+\beta} q^{\gamma} \tag{1.8}$$

(The former two relations are not independent of each other.) Further, for any two functions $u(q)$ and $v(q)$, whose fractional derivatives are defined, the following generalisation of the Liebnitz product rule holds (Osler, 1970):

$$\left(\frac{d}{dq}\right)^{\alpha} uv = \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{d}{dq}\right)^{\alpha-n} u . \left(\frac{d}{dq}\right)^{n} v \tag{1.9}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} = \alpha(\alpha-1)\dots(\alpha-n+1)$$

is the generalised binomial coefficient.

2. *Moyal Algebra with Fractional Powers*

According to (1.1),

$$p^{\alpha} \times q^{\beta} = e^{-i\partial^2/\partial p\partial q} p^{\alpha} q^{\beta}$$

Now by the Taylor expansion theorem, $\exp(\lambda\,\partial/\partial p)$ is a translation operator which takes $f(p)$ to $f(p+\lambda)$ so that

$$\begin{aligned} p^{\alpha} \times q^{\beta} &= (p - i\partial/\partial q)^{\alpha} q^{\beta} \\ &= \sum_{n=0}^{\infty} \binom{\alpha}{n} p^{\alpha-n} \left(-i\frac{\partial}{\partial q}\right)^{n} q^{\beta} \\ &= p^{\alpha} q^{\beta} \sum_{n=0}^{\infty} \binom{\alpha}{n} \binom{\beta}{n} n!\,(-ipq)^{-n} \end{aligned} \tag{2.1}$$

In the above steps the binomial theorem and (1.7) are used. By introducing a new function of the variable $z = -ipq$ by

$$S_{\alpha}^{\beta}(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \binom{\beta}{n} n!\, z^{-n} \tag{2.2}$$

the above product takes the form

$$p^{\alpha} \times q^{\beta} = p^{\alpha} q^{\beta} S_{\alpha}^{\beta}(-ipq) \tag{2.3}$$

We now use the Liebnitz product rule (1.9) with $v = z^{\beta}$ and in conjunction with (1.6) and (1.7):

$$\begin{aligned} \left(\frac{d}{dz}\right)^{\alpha} e^{z} z^{\beta} &= \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{d}{dz}\right)^{\alpha-n} e^{z} . \left(\frac{d}{dz}\right)^{n} z^{\beta} \\ &= e^{z} z^{\beta} \sum_{n=0}^{\infty} \binom{\alpha}{n} \binom{\beta}{n} n!\, z^{-n} \end{aligned}$$

This shows that S_α^β satisfies the equation

$$S_\alpha^\beta(z) = z^{-\beta} e^{-z} \left(\frac{d}{dz}\right)^\alpha e^z z^\beta \tag{2.4}$$

For positive integral values of α these are polynomials in z^{-1}. In this case they are related to the Laguerre polynomials by

$$(-z)^n S_n^{\beta+n}(-z) = n! L_n^\beta(z)$$

3. *Properties of S_α^β*

If α and β are both positive integers, we can obtain a double generating function

$$G(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S_n^m(z) \mu^m v^n / m! n!$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\mu^m/m!) z^{-m} e^{-z} e^{v\, d/dz} (e^z z^m)$$

Now $\exp(v\, d/dz).f(z) = f(z+v)$ so

$$G(z) = \sum_{m=0}^{\infty} (\mu^m/m!) e^v \left(1 + \frac{v}{z}\right)^m = e^v e^{\mu(1+v/z)}$$

That is,

$$G(z) = \exp(\mu + v + \mu v/z) \tag{3.1}$$

Thus $S_n^m(z)$ is the coefficient of $\mu^m v^n/m!n!$ in the expansion of

$$\exp(\mu + v + \mu v/z)$$

and we can write

$$S_n^m(z) = \left(\frac{d}{d\mu}\right)^m \left(\frac{d}{dv}\right)^n e^{\mu+v+\mu v/z}\Big|_{\mu=v=0} \tag{3.2}$$

We now show that (3.2) is valid in general for the S_α^β, that is,

$$S_\alpha^\beta(z) = \left(\frac{d}{d\mu}\right)^\beta \left(\frac{d}{dv}\right)^\alpha e^{\mu+v+\mu v/z}\Big|_{\mu=v=0} \tag{3.3}$$

with the definition (1.5) of fractional differentiation. (2.4) can be written

$$S_\alpha^\beta(z) = \left(\frac{d}{dz}\right)^\alpha \left(\frac{z}{y}\right)^\beta e^{z-y}\Big|_{y=z}$$

Applying the Liebnitz product rule,

$$S_\alpha^\beta(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{d}{dz}\right)^n \left(\frac{z}{y}\right)^\beta \cdot \left(\frac{d}{dz}\right)^{\alpha-n} e^{z-y}\Big|_{y=z}$$

$$= \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(\frac{d}{dv}\right)^n \left(1 + \frac{v}{y}\right)^\beta \cdot \left(\frac{d}{dv}\right)^{\alpha-n} e^v \Big|_{\substack{v=0\\y=z}}$$

(where $v = z - y$). Now $(d/dv)^{\beta-\alpha} e^v = e^v$ so that

$$S_\alpha^\beta(z) = \left(\frac{d}{dv}\right)^\alpha \left(1 + \frac{v}{z}\right)^\beta e^v \Big|_{v=0} \tag{3.4}$$

But

$$\left(1 + \frac{v}{z}\right)^\beta = \sum_{n=0}^\infty \binom{\beta}{n} \left(\frac{v}{z}\right)^n = \sum_{n=0}^\infty \binom{\beta}{n} \left(\frac{d}{d\mu}\right)^n e^{\mu v/z} \cdot \left(\frac{d}{d\mu}\right)^{-n} e^\mu \Big|_{\mu=0}$$

$$= (d/d\mu)^\beta e^{(\mu v/z)+\mu}\Big|_{\mu=0}$$

Substitution in (3.4) gives the required expression (3.3).

Two further expressions for S_α^β are easily obtained. From

$$S_\alpha^\beta(z) = z^{-\beta} \sum_{n=0}^\infty \binom{\alpha}{n} \left(\frac{d}{dz}\right)^n z^\beta$$

we have

$$S_\alpha^\beta(z) = z^{-\beta} \left(1 + \frac{d}{dz}\right)^\alpha z^\beta \tag{3.5}$$

Equation (3.5) can be written as

$$\left(1 + \frac{d}{dz}\right)^\alpha \left(\frac{z}{y}\right)^\beta \Big|_{z=y}$$

or, putting $z/y = \theta$,

$$S_\alpha^\beta(z) = \left(1 + \frac{1}{z}\frac{d}{d\theta}\right)^\alpha \theta^\beta \Big|_{\theta=1} \tag{3.6}$$

4. *Analytic Continuation of the Heisenberg Commutator*

We now consider the effect of operating with $\hat{p}^\alpha \hat{q}^\beta$ on a wave function $\psi(q)$ where $\hat{p} = -i d/dq$ and $\hat{q} = q$. This amounts to applying the product rule (1.9) to $q^\beta \psi(q)$. Thus,

$$\begin{aligned}
\hat{p}^\alpha \hat{q}^\beta \psi(q) &= e^{-i\pi\alpha/2} \left(\frac{d}{dq}\right)^\alpha q^\beta \psi(q) \\
&= e^{-i\pi\alpha/2} \sum_{n=0}^\infty \binom{\alpha}{n} \binom{\beta}{n} n!\, q^{\beta-n} \left(\frac{d}{dq}\right)^{\alpha-n} \psi(q) \\
&= \sum_{n=0}^\infty \binom{\alpha}{n} \binom{\beta}{n} n! (-i)^n \hat{q}^{\beta-n} \hat{p}^{\alpha-n} \psi(q)
\end{aligned} \tag{4.1}$$

which is precisely the operator obtained by substituting $p = \hat{p}$ and $q = \hat{q}$ in (2.3), with all $\hat{q}$'s to the left and $\hat{p}$'s to the right. Thus, the isomorphism between the Moyal and operator algebras exists for a much wider class of functions of p and q than power series; one can have any complex power of p and q.

It is now easy to get an expression for the commutator $[\hat p^\alpha, \hat q^\beta]$. The right side of (4.1) is

$$\left(\hat q^\beta \hat p^\alpha + \sum_{n=0}^{\infty}\binom{\alpha}{n}\binom{\beta}{n} n!(-i)^n \hat q^{\beta-n}\hat p^{\alpha-n}\right)\psi(q) = (\hat q^\beta \hat p^\alpha + \hat q^\beta \hat S'^{\beta}_{\alpha}(-iqp)\hat p^\alpha)\psi(q)$$

where S' denotes deletion of the first term in the expansion of S and $\hat S'$ is the corresponding operator with the $\hat q$'s to the left of the $\hat p$'s. Thus

$$\hat p^\alpha \hat q^\beta - \hat q^\beta \hat p^\alpha = \hat q^\beta \hat S'^{\beta}_{\alpha}(-iqp)\hat p^\alpha \tag{4.2}$$

is the analytic continuation of the Heisenberg commutator for arbitrary complex powers of the operators.

Further if $f(p)$ and $g(q)$ are any two functions, we can use the results of Section 2 to state that the operator $f(\hat p)g(\hat q)$ is obtained in standard order by evaluating

$$g(\hat q - i\partial/\partial\hat p)f(\hat p)$$

Equivalently, it can be written as

$$e^{-i\partial^2/\partial\hat p\partial\hat q}g(\hat q)f(\hat p)$$

The resulting expansions will, of course, be an infinite series, whereas for polynomial f and g we obtain a finite number of terms.

5. *The Generalised Correspondence Rule*

It is well known that there are various phase space distributions corresponding to different rules of transforming functions to operators. In Cohen (1966) general expressions for these distributions and operators are obtained and in Misra & Shankara (1968) the specific representation for this transformation is given. We now wish to generalise this rule for functions which contain arbitrary powers of p and q.

In this generalised rule, the operators corresponding to the monomial $p^m q^n$ is the coefficient of $\mu^m \nu^n/m!n!$ in the expansion of

$$\lambda(\mu, \nu)\, e^{\mu\hat p+\nu\hat q} \tag{5.1}$$

where λ is any power series subject to the restrictions

$$\lambda(\mu, 0) = \lambda(0, \nu) = 1 \tag{5.2}$$

Expressed in standard form the required coefficient in (5.1) is

$$\left(\frac{\partial}{\partial\mu}\right)^m\left(\frac{\partial}{\partial\nu}\right)^n \lambda(\mu, \nu)\exp\left[\nu\left(\hat q - \frac{i\mu}{2}\right)\right]\exp(\mu\hat p)|_{\mu=\nu=0} \tag{5.3}$$

In this form it is readily generalisable in two ways: one, to all complex powers of p and q and two, to any λ subject to the restrictions (5.2). Thus,

$$\left(\frac{\partial}{\partial\mu}\right)^\alpha\left(\frac{\partial}{\partial\nu}\right)^\beta \lambda(\mu, \nu)\exp\left[\nu\left(\hat q - \frac{i\mu}{2}\right)\right]\exp(\mu\hat p)|_{\mu=\nu=0} \tag{5.4}$$

is the operator corresponding to the product $p^\alpha q^\beta$ for all complex α and β. Also, since $\lambda(\mu,\nu)$ is not only a polynomial but can be any function for which fractional differentiation is defined, the rule (5.4) is more general.

It is desirable to extend (5.4) to find operators corresponding to any $f(p,q)$ for which fractional differentiation is defined. For this purpose, consider the expression

$$\lambda\left(\frac{\partial}{\partial p},\frac{\partial}{\partial q}\right)\exp\left(-\frac{i}{2}\partial^2/\partial p\,\partial q\right)p^\alpha q^\beta \tag{5.5}$$

This can be written as

$$\lambda\left(\frac{\partial}{\partial p},\frac{\partial}{\partial q}\right)\exp\left(-\frac{i}{2}\partial^2/\partial p\,\partial q\right)\left(\frac{\partial}{\partial \mu}\right)^\alpha\left(\frac{\partial}{\partial \nu}\right)^\beta \exp(\mu p+\nu q)|_{\mu=\nu=0}$$
$$=\left(\frac{\partial}{\partial \mu}\right)^\alpha\left(\frac{\partial}{\partial \nu}\right)^\beta \lambda(\mu,\nu)\exp(-i\mu\nu/2)\exp(\mu p+\nu q) \tag{5.6}$$

Comparing this expression with (5.4) we obtain the following result: The operator corresponding to $p^\alpha q^\beta$ is obtained by evaluating (5.5) and replacing p and q by $\hat{p}$ and $\hat{q}$ in standard order. More generally, the operator $\hat{f}(\hat{p},\hat{q})$ is obtained by replacing p by $\hat{p}$ and q by $\hat{q}$ in

$$\lambda\left(\frac{\partial}{\partial p},\frac{\partial}{\partial q}\right)\exp\left(-\frac{i}{2}\partial^2/\partial p\,\partial q\right)f(p,q) \tag{5.7}$$

after rearranging it in standard order.

6. *Conclusion*

In the literature, the Moyal algebra of functions on phase space is restricted to polynomials. The same restriction occurs in the discussion of Heisenberg commutators of canonical operators. This restriction has been lifted and both these algebras have been extended to include products of arbitrary complex powers of the conjugate variables. The isomorphism between Moyal and operator algebras is found to exist for a much wider class of functions than power series. The generalised correspondence rule of Cohen is shown to be capable of further generalisation using a non-polynomial parametric function.

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